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ON THE PROFINITE ABELIAN BECKMANN-BLACK PROBLEM

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Abstract. The main topic of this paper is to generalize the problem of Beckmann-Black for profinite groups. We introduce the Beckmann-Black problem for complete systems of finite groups and for unramified extensions. We prove that every Galois extension of profinite abelian group over a ψ-free field is the specialization of some tower of regular Galois extensions of the same group.

1. Presentation

1.1. **Notation and definitions.** Let K be a field and $(G_n, s_n)_{n\in\mathbb{N}}$ be a complete system, *i.e.*, a projective system of finite groups G_n ($n \in \mathbb{N}$) and epimorphisms $s_n : G_n \to G_{n-1}$ ($n > 0$).

An *abelian complete system* is a complete system $(G_n, s_n)_{n\in\mathbb{N}}$ such that G_n is an abelian group for every $n \in \mathbb{N}$.

Denote by $\overline{K}$ an algebraic closure of K and let G_K be the absolute Galois group of K. Denote by $K(T)$ the field of rational functions in one variable T with coefficients in K.

A finite extension $F/K(T)$ is said to be a regular Galois extension of group G if $F/K(T)$ is a Galois extension of group G and the function field F/K is regular. Recall that *regular* means $F \cap \overline{K} = K$.

In this paper, we want to generalize one of open questions in Inverse Galois Theory known as the Beckmann-Black problem. More precisely, *if K is a field and G is a finite group, then the Beckmann-Black problem asks whether each Galois extension E/K of group G is the specialization of some regular Galois extension $F/K(T)$ of group G at some unramified point $t_0 \in \mathbb{P}^1(K)$.*

The Beckmann-Black problem for finite groups is known to have a positive answer in some situations. For example:

- G is a symmetric group (Beckmann [Be] if K is a number field, Black [Bl2] for an arbitrary field).
- G is the dihedral group D_n of order $2n$ when n is odd (Black [Bl1]).
- G is an abelian group (Beckmann [Be] and Black [Bl1] if K is a number field, and Dèbes [D1] if K is an arbitrary field).

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- G is a finite group and K is P(seudo) A(lgebraically) C(losed)[1] (Dèbes [D1]).

Throughout this paper, we assume K is a perfect field.

1.2. **Profinite Beckmann-Black Problem.** Let K be a field and $(G_n, s_n)_{n\in\mathbb{N}}$ be a complete system. Fix a tower of Galois extensions $(E_n/K)_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$; this means that E_n/K is a Galois extension of group G_n ($n \geqslant 0$) such that ${E_n}^{\ker(s_n)} = E_{n-1}$ for every $n \geqslant 1$. The question of Profinite Beckmann-Black over K asks to find:

(i) a tower of regular Galois extensions $(F_n/K(T))_{n\in\mathbb{N}}$ realizing the complete system $(G_n, s_n)_{n\in\mathbb{N}}$.

(ii) an unramified point $t_0 \in \mathbb{P}^1(K)$ such that the specialization $(F_n)_{t_0}/K$ of $F_n/K(T)$ at this point t_0 is a Galois extension isomorphic to E_n/K for every $n \in \mathbb{N}$.

Note that the specialization of $F_n/K(T)$ is independent of the selected point over the point t_0. So we can define, without problem, the specialization of a tower of regular Galois extensions at the point t_0.

1.3. **Main result.** Our purpose in this paper is to study the Profinite Beckmann-Black Problem for abelian complete systems. We will prove, in §3, the following result.

Theorem 1.1. *Let K be an uncountable regular ψ-free field and let $(G_n, s_n)_{n\in\mathbb{N}}$ be an abelian complete system. For every tower of Galois extensions $(E_n/K)_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$, there exists a tower of regular Galois extensions $(F_n^E/K(T))_{n\in\mathbb{N}}$ (geometrically) unramified at a point $T = t_0 \in \mathbb{P}^1(K)$ with specialization $((F_n^E)_{t_0}/K(T))_{n\in\mathbb{N}}$ isomorphic to the tower $(E_n/K)_{n\in\mathbb{N}}$.*

A typical example of K satisfying the assumption of the above theorem is the field of formal Laurent series $\mathbb{Q}^{\mathrm{ab}}((x))$. See §2.1 below.

2. Preliminary reminders

2.1. **Regular ψ-free field.** A field K is said to be a *regular ψ-free* field if, for every complete system $(G_n, s_n)_{n\in\mathbb{N}}$, there exists a tower of regular Galois extensions $(F_n/K(T))_{n\in\mathbb{N}}$ realizing regularly the complete system $(G_n, s_n)_{n\in\mathbb{N}}$. This means that, for every $n \in \mathbb{N}$, there exists a regular Galois extension $F_n/K(T)$ of group G_n such that there exists an epimorphism

$$\varepsilon_n : \mathrm{Gal}(F_n/K(T)) \to G_n$$

[1] Recall that a field K is PAC if each geometrically irreducible variety V defined over K has infinitely many K-rational points.

making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gal}(F_{n+1}/K(T)) \arrow[r] \arrow[d, "\varepsilon_{n+1}"'] & \mathrm{Gal}(F_n/K(T)) \arrow[d, "\varepsilon_n"] \\
G_{n+1} \arrow[r, "s_{n+1}"] & G_n
\end{tikzcd}.$$

Moreover, if K is a henselian field of characteristic 0 such that $[K(\mu_\infty) : K] < \infty$, then K is a regular ψ-free field [DD2, theorem (3.4)].

2.2. **Specialization.** Let G be a finite group, K be a field and $F/K(T)$ be a regular Galois extension of group G with a K-rational branch divisor $\mathbf{t}$. This extension corresponds to some epimorphism $\phi : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K \to G$ and the extension $F\overline{K}/\overline{K}(T)$ corresponds to the restriction $\overline{\phi} : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}} \to G$ of ϕ to $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}}$; it is still surjective as F/K is a regular extension. Let $t_0 \in \mathbb{P}^1(K) \setminus \mathbf{t}$ be a K-rational point and consider the section, s_{t_0}, corresponding to this point.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}} \arrow[r] \arrow[d, "\overline{\phi}"'] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K \arrow[r] \arrow[d, "\phi"] & G_K \arrow[r] \arrow[l, bend right, "s_{t_0}"'] & 1. \\
& G \arrow[r, equal] & G & &
\end{tikzcd}$$

The specialization, F_{t_0}, of $F/K(T)$ at $T = t_0$ is the residue field of F at some prime above t_0 in the extension $F/K(T)$. The specialization F_{t_0} corresponds to the homomorphism $\phi \circ s_{t_0}$ ([D2, proposition(2.1)]). More precisely, F_{t_0} is the fixed field in $\overline{K}$ of $\ker(\phi \circ s_{t_0})$. In particular, the specialization F_{t_0}/K is a Galois field extension of group $\mathrm{Im}(\phi \circ s_{t_0}) \subset G$. The morphism $\phi \circ s_{t_0}$ is called the specialization morphism of $F/K(T)$ at t_0. For more details, we refer to [D2, chapter 3] and to [D1].

3. Proof of theorem 1.1

Let K be an uncountable regular ψ-free field and $(G_n, s_n)_{n\in\mathbb{N}}$ be an abelian complete system. Denote by $K(\!(T)\!)$ the field of formal Laurent series in T with coefficients in K.

To prove our theorem, we have three stages. Firstly, we show that our hypothesis implies that:

Proposition 3.1. *There exist a point $t_0 \in \mathbb{P}^1(K)$ and a tower of regular Galois extensions $(\widehat{F}_n/K(T))_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$ such that $\widehat{F}_n \subset K(\!(T - t_0)\!)$ for every $n \in \mathbb{N}$.*

For second stage, let $(H_n, \gamma_n)_{n\in\mathbb{N}}$ be a *sub-system* of $(G_n, s_n)_{n\in\mathbb{N}}$; this means that H_n is a sub-group of G_n and the restriction of s_n on H_n is γ_n, for each $n \in \mathbb{N}$. We will prove that Proposition 3.1 implies the following conclusion:

Proposition 3.2. *Let $(G_n, s_n)_{n\in\mathbb{N}}$ be an abelian complete system and $(H_n, \gamma_n)_{n\in\mathbb{N}}$ be a sub-system of $(G_n, s_n)_{n\in\mathbb{N}}$. For each tower of Galois extensions $(E_n/K)_{n\in\mathbb{N}}$ of group $(H_n, \gamma_n)_{n\in\mathbb{N}}$, there exists a tower of regular Galois extensions $(F_n^E/K(T))_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$ such that its specialization at the point $T = t_0$ is a tower of Galois extension of group $(H_n, \gamma_n)_{n\in\mathbb{N}}$ isomorphic to $(E_n/K)_{n\in\mathbb{N}}$.*

At third stage, taking $H_n = G_n$ in the proposition 3.2 gives the conclusion of our result.

• 1^{st} **stage "Proof of Proposition 3.1"**

As K is a regular ψ-free field, there exists a tower of regular Galois extensions $(L_n/K(T))_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$. Since there are at most countably many branch points in $\bigcup_n L_n/K(T)$, there is an unramified point $T = t_0 \in \mathbb{P}^1(K)$ in the function field tower $(L_n/K(T))_{n\in\mathbb{N}}$.

Denote by $\phi_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_n$ the regular representation corresponding to the regular Galois extension $L_n/K(T)$, where t_n is the branch point set of this extension. Let $r_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_K$ be the natural restriction and $s_{t_0,n} : G_K \to \pi_1(\mathbb{P}^1 \setminus t_n)_K$ the section corresponding to the point $T = t_0$.

On the other hand, for each $n \in \mathbb{N}$, we have $t_{n-1} \subseteq t_n$. So there exists a natural morphism $i_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to \pi_1(\mathbb{P}^1 \setminus t_{n-1})_K$ such that $i_n(x) = 1$ for every $x \in t_n \setminus t_{n-1}$. The regular Galois extension $L_{n-1}/K(T)$ is an unramified extension over $t_n \setminus t_{n-1}$, so the morphism $s_n \circ \phi_n$ factors through i_n to give $s_n \circ \phi_n = \phi_{n-1} \circ i_n$ for every $n \in \mathbb{N}$. Furthermore, we have $r_n = r_{n-1} \circ i_n$ and $i_n \circ s_{t_0,n} = s_{t_0,n-1}$.

Fix $n \in \mathbb{N}$. Let $\widehat{\phi}_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_n$ be a map defined as follows: for $x \in \pi_1(\mathbb{P}^1 \setminus t_n)_K$, we pose

$$\widehat{\phi}_n(x) = \phi_n(x).(\phi_n \circ s_{t_0,n} \circ r_n(x))^{-1}.$$

This map $\widehat{\phi}_n$ is a group homomorphism (because G_n is an abelian group) with image group equal to G_n. Thus $\widehat{\phi}_n$ defines a regular Galois extension $\widehat{F}_n/K(T)$ of group G_n. Furthermore, $\widehat{\phi}_n$ and ϕ_n coincide over $\overline{K}(T)$. this implies that

$$L_n\overline{K} = \widehat{F}_n\overline{K}.$$

For every $n \in \mathbb{N}$, we have:

$$s_n \circ \widehat{\phi}_n = s_n \circ (\phi_n.(\phi_n \circ s_{t_0,n} \circ r_n)^{-1})$$

$$
\begin{aligned}
&= (s_n \circ \phi_n).(s_n \circ \phi_n \circ s_{t_0,n} \circ r_n)^{-1} \\
&= (\phi_{n-1} \circ i_n).(\phi_{n-1} \circ i_n \circ s_{t_0,n} \circ r_n)^{-1} \\
&= (\phi_{n-1} \circ i_n).(\phi_{n-1} \circ s_{t_0,n-1} \circ r_{n-1} \circ i_n)^{-1} \\
&= ((\phi_{n-1}).(\phi_{n-1} \circ s_{t_0,n-1} \circ r_{n-1})^{-1}) \circ i_n = \widehat{\phi}_{n-1} \circ i_n.
\end{aligned}
$$

Thus $s_n \circ \widehat{\phi}_n = \widehat{\phi}_{n-1} \circ i_n$. This implies that the extension $\widehat{F}_n/K(T)$, $n \in \mathbb{N}$, form a tower of regular Galois extensions of group $(G_n, s_n)_{n\in\mathbb{N}}$. Furthermore, we have $\widehat{\phi}_n \circ s_{t_0,n} = (\phi_n \circ s_{t_0,n}).(\phi_n \circ s_{t_0,n})^{-1} = 1$, hence $\widehat{F}_n \subseteq K((T - t_0))$ for every $n \in \mathbb{N}$.

• **2^{nd} stage "Proof of Proposition 3.2".**

Let $(H_n, \gamma_n)_{n\in\mathbb{N}}$ be any sub-system of $(G_n, s_n)_{n\in\mathbb{N}}$. Suppose given a tower of Galois extensions $(E_n/K)_{n\in\mathbb{N}}$ of group $(H_n, \gamma_n)_{n\in\mathbb{N}}$. By virtue of Proposition 3.1, we find a tower of regular Galois extension $(\widehat{F}_n/K(T))_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$ such that $\widehat{F}_n \subseteq K((T - t_0))$ for some unramified point $t_0 \in K$. We want to replace the above tower $(\widehat{F}_n/K(T))_{n\in\mathbb{N}}$ by another tower $(F_n^E/K(T))_{n\in\mathbb{N}}$ of regular Galois extensions of group $(G_n, s_n)_{n\in\mathbb{N}}$ so that its specialization at the point $T = t_0$ is a tower of Galois extension of group $(H_n, \gamma_n)_{n\in\mathbb{N}}$ isomorphic to $(E_n/K)_{n\in\mathbb{N}}$.

We give two arguments. The first one uses a similar process as in first stage. The second one is essentially equivalent but uses a different formalism.

[*Argument 1.*] Consider the representation $\widehat{\phi}_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_n$ corresponding to the regular Galois extension $\widehat{F}_n/K(T)$. Recall that $L_n\overline{K} = \widehat{F}_n\overline{K}$, in particular the branch point set of $\widehat{F}_n/K(T)$ is t_n and t_0 is unramified.

Still denote by $r_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_K$ the natural restriction, by $s_{t_0,n} : G_K \to \pi_1(\mathbb{P}^1 \setminus t_n)_K$ the section corresponding to the point t_0 and by $i_n : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to \pi_1(\mathbb{P}^1 \setminus t_{n-1})_K$ the natural morphism given by $t_{n-1} \subseteq t_n$. Recall that $s_n \circ \phi_n = \phi_{n-1} \circ i_n$, $r_n = r_{n-1} \circ i_n$ and $i_n \circ s_{t_0,n} = s_{t_0,n-1}$, for every $n > 0$.

Let $\varphi_n : \mathrm{G}_K \to H_n \subset G_n$ be a representation of the Galois extension E_n/K ($n \in \mathbb{N}$); we have $E_n = (\overline{K})^{\mathrm{Ker}(\varphi_n)}$ and $\gamma_n \circ \varphi_n = \varphi_{n-1}$.

Fix $n \in \mathbb{N}$. Let $\phi_n^E : \pi_1(\mathbb{P}^1 \setminus t_n)_K \to G_n$ be the map defined as follows: $\phi_n^E(x) = \widehat{\phi}_n(x).(\varphi_n \circ r_n(x))^{-1}$.

The map ϕ_n^E is a group homomorphism (because G_n is abelian) with image group equal to G_n, and coincides with $\widehat{\phi}_n$ and so with ϕ_n as well on $\pi_1(\mathbb{P}^1 \setminus t_n)_{\overline{K}}$. Thus ϕ_n^E defines a regular Galois extension $F_n^E/K(T)$ of group G_n and such that $F_n^E\overline{K} = F_n\overline{K}$.

For every $n > 0$, we have:

$$\begin{aligned} s_n \circ \phi_n^E &= s_n \circ (\widehat{\phi}_n.(\varphi_n \circ r_n)^{-1}) \\ &= (s_n \circ \widehat{\phi}_n).(\gamma_n \circ \varphi_n \circ r_n)^{-1} \\ &= (\widehat{\phi}_{n-1} \circ i_n).(\varphi_{n-1} \circ r_{n-1} \circ i_n)^{-1} \\ &= (\widehat{\phi}_{n-1}.(\varphi_{n-1} \circ r_{n-1})^{-1}) \circ i_n \\ &= \phi_{n-1}^E \circ i_n. \end{aligned}$$

Thus $s_n \circ \phi_n^E = \phi_{n-1}^E \circ i_n$. This implies that the extensions $F_n^E/K(T)$ ($n \in \mathbb{N}$) form a tower of regular Galois extensions of group $(G_n, s_n)_{n\in\mathbb{N}}$. Furthermore, we have $\phi_n^E \circ s_{t_0,n} = 1.\varphi_n^{-1} = \varphi_n^{-1}$ and so the specialized extension of $F_n^E/K(T)$ at t_0 is $\overline{K}^{\ker(\varphi_n^{-1})} = E_n$ ($n \in \mathbb{N}$).

Remark 3.3. Stage 1 and stage 2 could have been merged by defining directly ϕ_n^E in terms of ϕ_n as follows: for $x \in \pi_1(\mathbb{P}^1 \setminus t_n)_K$,

$$\phi_n^E(x) = \phi_n(x) \cdot (\phi_n \circ s_{t_0,n} \circ r_n(x))^{-1} \cdot (\varphi_n \circ r_n(x))^{-1} \quad (n \in \mathbb{N}).$$

[*Argument 2.*] Fix $n \in \mathbb{N}$. Denote by $\varphi_{n,t_0} : \widehat{F}_n \to K$ the K-place associated to $\widehat{F}_n$ at the point t_0. By an extension of scalars from K to E_n, the extension $\widehat{F}_n E_n/E_n(T)$ is a regular Galois extension of group G_n. As E_n/K is an algebraic extension, the point t_0 is still unramified in the extension $\widehat{F}_n E_n/K(T)$: $E_n\widehat{F}_n \subset E_n((T-t_0)) \subset \overline{K}((T-t_0))$.

On the other hand, $\widehat{F}_n/K(T)$ and E_n/K are two Galois extensions, so $\widehat{F}_n E_n/K(T)$ is a Galois extension of group isomorphic to $G_n \times H_n$, for every $n \in \mathbb{N}$. Consider the map $\rho_n : G_n \times H_n \to G_n$ given by $\rho_n(g,h) = gh$. This map ρ_n is a group homomorphism because G_n is an abelian group ($n \geqslant 0$).

Fix $n \in \mathbb{N}$. Denote by F_n^E the subfield of $E_n\widehat{F}_n$ fixed by $\mathrm{Ker}\rho_n$. Thus $F_n^E/K(T)$ is a Galois extension of group G_n. First we prove that F_n^E is a regular extension over K. This means that we must verify that $[F_n^E : K(T)] = [F_n^E\overline{K} : \overline{K}(T)]$. In fact, the two extensions $F_n^E/K(T)$ and $E_n(T)/K(T)$ are linearly disjoint because

$$\begin{aligned} F_n^E \cap E_n(T) &= (E_n\widehat{F}_n)^{\mathrm{Ker}\rho_n} \cap (E_n\widehat{F}_n)^{G_n} \\ &= (E_n\widehat{F}_n)^{\mathrm{Ker}\rho_n.G_n} = (E_n\widehat{F}_n)^{G_n \times H_n} = K(T). \end{aligned}$$

We deduce that $[F_n^E : K(T)] = [F_n^E E_n : E_n(T)]$. But the field $F_n^E E_n = \widehat{F}_n E_n$ is regular over E_n, so $[F_n^E E_n : E_n(T)] = [F_n^E\overline{E_n} : \overline{E_n}(T)]$. As E_n/K is a finite extension, then $\overline{E_n} = \overline{K}$. We conclude that

$$[F_n^E : K(T)] = [F_n^E E_n : E_n(T)] = [F_n^E\overline{K} : \overline{K}(T)].$$

Hence the finite extension $F_n^E/K(T)$ is a regular Galois extension of group G_n.

Furthermore, the following diagram:

$$\begin{tikzcd} G_n \times H_n \arrow[r, "\rho_n"] \arrow[d, "(s_n,\gamma_n)"'] & G_n \arrow[d, "s_n"] \\ G_{n-1} \times H_{n-1} \arrow[r, "\rho_{n-1}"'] & G_{n-1} \end{tikzcd}$$

is a commutative diagram because the restriction of s_n on H_n is equal to γ_n $(n > 0)$. We deduce that the family of regular Galois extensions $(F_n^E/K(T))_{n\in\mathbb{N}}$ form a tower of regular Galois extensions of group $(G_n, s_n)_{n\in\mathbb{N}}$.

Now we must show that the specialization of $(F_n^E/K(T))_{n\in\mathbb{N}}$ at the point $T = t_0$ is isomorphic to $(E_n/K)_{n\in\mathbb{N}}$.

Let us fix $n \in \mathbb{N}$ and we will study the specialization of $F_n^E/K(T))$ at the point $T = t_0$. Firstly, as $E_n \subseteq E_n\widehat{F}_n$, so $\varphi_{t_0}(E_n) \subseteq \varphi_{t_0}(E_n\widehat{F}_n)$. Now E_n/K is an extension geometrically unramified at t_0, so $\varphi_{t_0}(E_n) = E_n$. Thus $E_n \subseteq \varphi_{t_0}(E_n\widehat{F}_n)$. Denote by D_{t_0} the decomposition group of t_0 in $E_n\widehat{F}_n/K$. As $E_n \subseteq \varphi_{t_0}(E_n\widehat{F}_n)$, so $|D_{t_0}| = [\varphi_{t_0}(E_n\widehat{F}_n) : K] \geqslant [E_n : K] = |H_n|$.

Furthermore, we know that $(E_n\widehat{F}_n)^{H_n} = \widehat{F}_n$, so $\varphi_{t_0}((E_n\widehat{F}_n)^{H_n}) = \varphi_{t_0}(\widehat{F}_n)$. Now φ_{t_0} being a K-place means that $\varphi_{t_0}(\widehat{F}_n) = K$. Thus $\varphi_{t_0}((E_n\widehat{F}_n)^{H_n}) = K$.

As the point t_0 is unramified in $\widehat{F}_nE_n/K(T)$, so denote by $(E_n\widehat{F}_n)^{D_{t_0}}$ the subfield of $E_n\widehat{F}_n$ fixed by D_{t_0}. This field $(E_n\widehat{F}_n)^{D_{t_0}}$ is the biggest subfield of $E_n\widehat{F}_n$ such that $\varphi_{t_0}((E_n\widehat{F}_n)^{D_{t_0}}) = K$.

Indeed $\varphi_{t_0}((E_n\widehat{F}_n)^{H_n}) = K$, then $(E_n\widehat{F}_n)^{H_n} \subseteq (E_n\widehat{F}_n)^{D_{t_0}}$. Thus

$$[\widehat{F}_nE_n : (E_n\widehat{F}_n)^{D_{t_0}}] \leqslant [\widehat{F}_nE_n : (E_n\widehat{F}_n)^{H_n}] = |H_n|.$$

Thus

$$[\varphi_{t_0}(\widehat{F}_nE_n) : \varphi_{t_0}((E_n\widehat{F}_n)^{D_{t_0}})] \leqslant |H_n|$$

so $[\varphi_{t_0}(\widehat{F}_nE_n) : K] \leqslant |H_n|$.

We deduce that $|D_{t_0}| = [\varphi_{t_0}(\widehat{F}_nE_n) : K] = |H_n| = [E_n : K]$ and $E_n \subseteq \varphi_{t_0}(\widehat{F}_nE_n)$. Thus $E_n = \varphi_{t_0}(\widehat{F}_nE_n)$, so the specialization of $\widehat{F}_nE_n/K(T)$ at the point t_0 is isomorphic to E_n/K and $D_{t_0} = \mathrm{Gal}(E_n/K) = H_n$.

Finally, denote by $\widehat{\varphi_{t_0}}$ the restriction of φ_{t_0} on F_n^E. The specialization of $F_n^E/K(T)$ at the point t_0 is an intermediate extension of E_n/K (the specialization extension of $\widehat{F}_nE_n/K(T)$) of group equal to $\rho_n(D_{t_0})$. But

$\rho_n(D_{t_0}) = \rho_n(H_n) = \rho_n(\{1\} \times H_n) = H_n$. This implies that the specialization extension of $F_n^E/K(T)$ at the point t_0 is a Galois extension of group H_n isomorphic to E_n/K.

To sum up, we find a tower of regular Galois extensions $(F_n^E/K(T))_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$ such that the tower of specialization at the point $T = t_0$ is a Galois tower of group $(H_n, \gamma_n)_{n\in\mathbb{N}}$ isomorphic to $(E_n/K)_{n\in\mathbb{N}}$.

• **3^{rd} stage "Conclusion".**

Putting $G_n = H_n$ for every $n \in \mathbb{N}$ in Proposition 3.2, we conclude that, for each tower of Galois extensions $(E_n/K)_{n\in\mathbb{N}}$ of group $(G_n, s_n)_{n\in\mathbb{N}}$, there exists a tower of regular Galois extensions $(F_n^E/K(T))_{n\in\mathbb{N}}$ such that its specialization at the point $T = t_0$ is a tower of Galois extension isomorphic to $(E_n/K)_{n\in\mathbb{N}}$.

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References

[Be] S. Beckmann, *Is every extension of $\mathbb{Q}$ the specialization of a branched covering?*, J. Algebra **164** (1994), 430–451.

[Bl1] E. Black, *Arithmetic lifting of dihedral extensions*, J. Algebra **203** (1998), 12–29.

[Bl2] E. Black, *Deformations of Dihedral 2-group extensions of fields*, Trans. Amer. Math. Soc. **351** (1999), 3229–3241.

[D1] P. Dèbes, *Galois Covers with Prescribed Fibers: The Beckmann-Black problem*, Ann. Scuola Norm. Sup. Paris, Cl. Sci. **28** (1999), 273-286.

[D2] P. Dèbes, *Arithmétique des revêtements de la droite.*, Cours DEA (chapters 1-8), Univ. de Lille 1, 2009; math.univ-lille1.fr/∼pde/.

[DD2] P. Dèbes and B. Deschamps, *Corps ψ-libres et théorie inverse de Galois infinie* , J. Reine Angew. Math., **574** (2004), 197–218.

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